The validity of the following two equations can be verified by direct substitution of the preceding results:

$$P_1 m_1 + P_2 m_2 = m_G \tag{10.3-10}$$

and

$$P_1 + P_2 = 1 \tag{10.3-11}$$

where we have omitted the *k*s temporarily in favor of notational clarity.

In order to evaluate the "goodness" of the threshold at level k we use the normalized, dimensionless metric

$$\eta = \frac{\sigma_B^2}{\sigma_G^2} \tag{10.3-12}$$

where σ_G^2 is the *global variance* [i.e., the intensity variance of all the pixels in the image, as given in Eq. (3.3-19)],

$$\sigma_G^2 = \sum_{i=0}^{L-1} (i - m_G)^2 p_i \tag{10.3-13}$$

and σ_B^2 is the *between-class variance*, defined as

$$\sigma_B^2 = P_1(m_1 - m_G)^2 + P_2(m_2 - m_G)^2 \tag{10.3-14}$$

This expression can be written also as

$$\begin{aligned} \sigma_B^2 &= P_1 P_2 (m_1 - m_2)^2 \\ &= \frac{(m_G P_1 - m)^2}{P_1(1 - P_1)} \end{aligned} \tag{10.3-15}$$

The second step in Eq. (10.3-15) makes sense only if P_1 is greater than 0 and less than 1, which, in view of Eq. (10.3-11), implies that P_2 must satisfy the same condition.

where m_G and m are as stated earlier. The first line of this equation follows from Eqs. (10.3-14), (10.3-10), and (10.3-11). The second line follows from Eqs. (10.3-5) through (10.3-9). This form is slightly more efficient computationally because the global mean, m_G, is computed only once, so only two parameters, m and P_1, need to be computed for any value of k.

We see from the first line in Eq. (10.3-15) that the farther the two means m_1 and m_2 are from each other the larger σ_B^2 will be, indicating that the between-class variance is a measure of *separability* between classes. Because σ_G^2 is a constant, it follows that η also is a measure of separability, and maximizing this metric is equivalent to maximizing σ_B^2. The objective, then, is to determine the threshold value, k, that maximizes the between-class variance, as stated at the beginning of this section. Note that Eq. (10.3-12) assumes implicitly that $\sigma_G^2 > 0$. This variance can be zero only when all the intensity levels in the image are the same, which implies the existence of only one class of pixels. This in turn means that $\eta = 0$ for a constant image since the separability of a single class from itself is zero.

Reintroducing k, we have the final results:

$$\eta(k) = \frac{\sigma_B^2(k)}{\sigma_G^2} \tag{10.3-16}$$

and

$$\sigma_B^2(k) = \frac{\left[m_G P_1(k) - m(k)\right]^2}{P_1(k)\left[1 - P_1(k)\right]} \tag{10.3-17}$$

Then, the optimum threshold is the value, k^*, that maximizes $\sigma_B^2(k)$:

$$\sigma_B^2(k^*) = \max_{0 \le k \le L-1} \sigma_B^2(k) \tag{10.3-18}$$

In other words, to find k^* we simply evaluate Eq. (10.3-18) for all *integer* values of k (such that the condition $0 < P_1(k) < 1$ holds) and select that value of k that yielded the maximum $\sigma_B^2(k)$. If the maximum exists for more than one value of k, it is customary to average the various values of k for which $\sigma_B^2(k)$ is maximum. It can be shown (Problem 10.33) that a maximum always exists, subject to the condition that $0 < P_1(k) < 1$. Evaluating Eqs. (10.3-17) and (10.3-18) for all values of k is a relatively inexpensive computational procedure, because the maximum number of integer values that k can have is L.

Once k^* has been obtained, the input image $f(x, y)$ is segmented as before:

$$g(x, y) = \begin{cases} 1 & \text{if } f(x, y) > k^* \\ 0 & \text{if } f(x, y) \le k^* \end{cases} \tag{10.3-19}$$

for $x = 0, 1, 2, \ldots, M - 1$ and $y = 0, 1, 2, \ldots, N - 1$. Note that all the quantities needed to evaluate Eq. (10.3-17) are obtained using only the histogram of $f(x, y)$. In addition to the optimum threshold, other information regarding the segmented image can be extracted from the histogram. For example, $P_1(k^*)$ and $P_2(k^*)$, the class probabilities evaluated at the optimum threshold, indicate the portions of the areas occupied by the classes (groups of pixels) in the thresholded image. Similarly, the means $m_1(k^*)$ and $m_2(k^*)$ are estimates of the average intensity of the classes in the original image.

The normalized metric η, evaluated at the optimum threshold value, $\eta(k^*)$, can be used to obtain a quantitative estimate of the separability of classes, which in turn gives an idea of the ease of thresholding a given image. This measure has values in the range

Although our interest is in the value of η at the optimum threshold, k^*, this inequality holds in general for any value of k in the range $[0, L - 1]$.

$$0 \le \eta(k^*) \le 1 \tag{10.3-20}$$

The lower bound is attainable only by images with a single, constant intensity level, as mentioned earlier. The upper bound is attainable only by 2-valued images with intensities equal to 0 and $L - 1$ (Problem 10.34).